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projectile if the velocities and deviations of the axes are equal. The mass of the shot, $\propto d^3$; moment of inertia $\propto d^5$; resistance $\propto d^2$; tilting couple $\propto d^3$; hence the velocity of precession, which varies as the tilting couple ÷ moment of inertia, $\propto d^{-2}$.

The curvature of the path due to the normal component of the resistance $\propto$ resistance ÷ momentum of shot $\propto d^2/d^3 \propto d^{-1}$. Radius of the helix $\propto$ curvature of the path ÷ velocity of precession $\propto d^{-1} \times d^2 \propto d$. The radius of the helix of a 0·3-inch bullet would therefore be of the order of $0{\cdot}4''/40 = 0{\cdot}01$ inch, which would not be detectable on the screen records.]

On the Graphical Determination of Fresnel's Integrals.

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The functions $\int_0^x \cos(\frac{1}{2}\pi x^2)\,dx$ and $\int_0^x \sin(\frac{1}{2}\pi x^2)\,dx$, known as Fresnel's integrals, have usually been evaluated by means of converging series, for instance those of Cauchy and Knochenhauer. The graphical method of integration, by the summation of the areas contained between given ordinates, the arc of the curve they cut off and the x-axis, is readily applicable in this case, the quantity $\cos\frac{1}{2}\pi x^2$ (or $\sin\frac{1}{2}\pi x^2$) being plotted as ordinate against x as abscissa. The area can then be determined between any given limits by Simpson's Rule. In practice, however, it is simpler and more accurate to apply Simpson's Rule directly to the calculated values of the ordinates, without plotting a curve.

The curve $y = \cos\frac{1}{2}\pi x^2$ has its zero values at $x = 1, 3^{\frac{1}{2}}, 5^{\frac{1}{2}}$, etc., and is thus marked off into a series of loops of steadily decreasing area. The simplest method of computation is to find the areas of these separate loops (the first being only half a loop) ranging, in values of x, from 0 to 1, 1 to $3^{\frac{1}{2}}$, $3^{\frac{1}{2}}$ to $5^{\frac{1}{2}}$, etc. The range for a particular loop will be termed its base-line.

In the subjoined tables the calculations have been made by dividing each base-line into 10 parts and calculating the ordinates corresponding to the points of division, *e.g.*, 0·1, 0·2, 0·3...0·9 for the first half-loop.

The area of any loop may then be written as kd, where d is the length of the base-line and k a factor depending upon the loop considered. The

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values of k are shown in Table I for the first six loops and for two other loops of higher order. The table shows that k rapidly approaches a constant limiting value, 0·6366. Thus after a very few loops the area of a loop is at once given by the product of this constant and the length of the base-line. The theoretical limit of k is discussed in Section 4.

The integration $\int_0^x \cos\left(\frac{1}{2}\pi x^2\right) dx$ to any upper limit $x = n^{\frac{1}{2}}$, where n is an odd integer, is thus readily performed by adding to the integral for the first few loops (say up to $11^{\frac{1}{2}}$) the value for the higher loops $k\Sigma d$, where Σd represents the quantity obtained by summing up the base-lines, attributing to them positive and negative values alternately (because of the alternating signs of the loop-areas), and writing $k = 0{\cdot}6366$.

Table I exhibits also the areas of the first six loops, and so provides the data for such a calculation.

The values of $\int_0^x \sin\left(\frac{1}{2}\pi x^2\right) dx$ can similarly be calculated for the loop-areas of $y = \sin \frac{1}{2}\pi x^2$, where the base-lines range from 0 to $2^{\frac{1}{2}}$, $2^{\frac{1}{2}}$ to 2, 2 to $6^{\frac{1}{2}}$, etc., the values of x for which $\sin \frac{1}{2}\pi x^2 = 0$.

These loop-areas and the corresponding values of k are shown in Table II; we again find the rapid approach of k to the constant value 0·6366.

Table I.—Constants and Areas of Loops of $y = \cos \frac{1}{2}\pi x^2$.

Limits.	Base-line.	Area.	k.
0—1	1·0000	0·7799	0·7799
1—$3^{\frac{1}{2}}$	0·7320	0·4589	0·6268
$3^{\frac{1}{2}}$—$5^{\frac{1}{2}}$	0·5040	0·3198	0·6344
$5^{\frac{1}{2}}$—$7^{\frac{1}{2}}$	0·4097	0·2603	0·6353
$7^{\frac{1}{2}}$—3	0·3542	0·2254	0·6362
3—$11^{\frac{1}{2}}$	0·3166	0·2015	0·6363
$99^{\frac{1}{2}}$—$101^{\frac{1}{2}}$	0·1000	—	0·6365
$999^{\frac{1}{2}}$—$1001^{\frac{1}{2}}$	0·0316	—	0·6366

Table II.—Constants and Area of Loops of $y = \sin \frac{1}{2}\pi x^2$.

Limits.	Base-line.	Area.	k.
0—$2^{\frac{1}{2}}$	1·4142	0·7145	0·5052
$2^{\frac{1}{2}}$—2	0·5858	0·3706	0·6327
2—$6^{\frac{1}{2}}$	0·4495	0·2856	0·6353
$6^{\frac{1}{2}}$—$8^{\frac{1}{2}}$	0·3789	0·2409	0·6357
$8^{\frac{1}{2}}$—$10^{\frac{1}{2}}$	0·3339	0·2124	0·6362
$10^{\frac{1}{2}}$—$12^{\frac{1}{2}}$	0·3018	0·1921	0·6363

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2. Integration can of course be performed to any upper limit x by applying Simpson's Rule to the portion of a loop between $n^{\frac{1}{2}}$ and x, where n is the largest odd integer less than x^2 in the case of $\cos\frac{1}{2}\pi x^2$, the largest even integer less than x^2 in the case of $\sin\frac{1}{2}\pi x^2$. A simpler method is given in Section 5, for cases in which n is not small.

Table III gives the values of $\int_0^x \cos(\frac{1}{2}\pi x^2)\,dx$ for $x = 0{\cdot}1$, $0{\cdot}2$, etc., … 1, calculated without further subdivision of the base-line of the first half loop, Gilbert's values being added for comparison. It is interesting to note, with a view to the availability of this procedure in cases where other methods may not be available, that such slight subdivision can give satisfactory results.

Table III.

Upper limit x.	$\int_0^x \cos(\frac{1}{2}\pi x^2)dx$.	Gilbert's table.
0·1	0·1000	0·0999
0·2	0·1999	0·1999
0·3	0·2994	0·2994
0·4	0·3975	0·3975
0·5	0·4923	0·4923
0·6	0·5811	0·5811
0·7	0·6597	0·6597
0·8	0·7229	0·7230
0·9	0·7648	0·7648
1·0	0·7799	0·7799

3. The values of $\int_{x_1}^{x_2}\cos(\frac{1}{2}\pi x^2)\,dx$ and $\int_{x_1}^{x_2}\sin(\frac{1}{2}\pi x^2)\,dx$ in the special case when x_1 and x_2 are not small and differ but little from each other, can be directly found as follows:—

$$d(x^2) = 2x\,dx \quad \text{and} \quad 2x = x_1+x_2, \text{ approximately.}$$

So, approximately,

$$\cos(\tfrac{1}{2}\pi x^2)\,dx = \frac{1}{x_1+x_2}\cos(\tfrac{1}{2}\pi x^2)\,d(x^2),$$

whence
$$\int_{x_1}^{x_2}\cos(\tfrac{1}{2}\pi x^2)\,dx = \frac{2}{\pi(x_1+x_2)}(\sin\tfrac{1}{2}\pi x_2^2 - \sin\tfrac{1}{2}\pi x_1^2). \qquad (1)$$

Similarly,

$$\int_{x_1}^{x_2}\sin(\tfrac{1}{2}\pi x^2)\,dx = \frac{2}{\pi(x_1+x_2)}(\cos\tfrac{1}{2}\pi x_1^2 - \cos\tfrac{1}{2}\pi x_2^2). \qquad (2)$$

4. Special cases.—(*a*) $\int_{x_1}^{x_2} \cos\left(\tfrac{1}{2}\pi x^2\right) dx$.

In the case when $x_1^2 = 2n-1$, where n is any integer not too small for the approximation of Section 3, the integral becomes $\dfrac{2}{\pi(x_1+x_2)}\left(\sin \tfrac{1}{2}\pi x_2^2 \pm 1\right)$, where the positive sign refers to even values of n, the negative to odd values. If, in addition, $x_2^2 = 2n+1$, the value reduces to $\pm \dfrac{4}{\pi(x_1+x_2)}$, or, as in this case $x_2^2 - x_1^2 = 2$, to

$$\pm \frac{2(x_2-x_1)}{\pi} = \pm\, 0{\cdot}6366\,(x_2-x_1). \tag{3}$$

This expression therefore gives the area of the loop of the curve comprised between the limits $x_1 = (2n-1)^{\frac{1}{2}}$ and $x_2 = (2n+1)^{\frac{1}{2}}$, and is numerically equal to 0·6366 of the base-line of the loop.

(*b*) $\int_{x_1}^{x_2} \sin\left(\tfrac{1}{2}\pi x^2\right) dx$. A similar investigation shows that if $x_1^2 = 2n$, the value becomes $\dfrac{2}{\pi(x_1+x_2)}\left(\pm 1 - \cos \tfrac{1}{2}\pi x_2^2\right)$, where the positive sign is to be associated as before with even values of n.

If we consider one loop, *i.e.* if $x_2 = 2n+2$, the integral becomes again $2(x_2-x_1)/\pi$, showing that the limiting value of k is again 0·6366.

5. $\int_0^x \cos\left(\tfrac{1}{2}\pi x^2\right) dx = \int_0^{\sqrt{n}} \cos\left(\tfrac{1}{2}\pi x^2\right) dx + \int_{\sqrt{n}}^{x} \cos\left(\tfrac{1}{2}\pi x^2\right) dx$, where n is the greatest odd integer less than x^2.

From equations (1) and (3) computations may now be made rapidly; thus, for example:—

$$\int_0^{3\cdot 1} \cos\left(\tfrac{1}{2}\pi x^2\right) dx = \int_0^{3} \cos\left(\tfrac{1}{2}\pi x^2\right) dx + \int_3^{3\cdot 1} \cos\left(\tfrac{1}{2}\pi x^2\right) dx$$

$$= 0{\cdot}7799 - 0{\cdot}4589 + 0{\cdot}3198 - 0{\cdot}2603 + 0{\cdot}2254 + \frac{2\left(\sin \frac{1}{2}(3{\cdot}1)^2\pi - 1\right)}{\pi(3{\cdot}0 + 3{\cdot}1)}.$$

The last quantity $= -0{\cdot}0444$, so the integral is 0·5615.

$\int_0^x \sin\left(\tfrac{1}{2}\pi x^2\right) dx$ can be computed in a similar way, the integer corresponding to n being even instead of odd.

Table IV shows the values of the integral $\int_0^x \cos\left(\tfrac{1}{2}\pi x^2\right) dx$ calculated by the above method for upper limits ranging from 3·0 to 3·5 by steps of 0·1.

Table IV.

Upper limit x.	$\int_0^x \cos\left(\frac{1}{2}\pi x^2\right)dx$.	Gilbert's table.
3·0	0·6059	0·6057
3·1	0·5615	0·5616
3·2	0·4662	0·4663
3·3	0·4058	0·4057
3·4	0·4386	0·4385
3·5	0·5327	0·5326

6. The method is also capable of indicating the values (0·5) of the integrals between the limits 0 and ∞. Thus

$$\int_0^\infty \sin\left(\tfrac{1}{2}\pi x^2\right) dx = \int_0^{\sqrt{(n_0-2)}} y\,dx + \int_{\sqrt{(n_0-2)}}^\infty y\,dx, \qquad (4)$$

where $n_0 = 4\text{N}$, N being any integer.

$$= \int_0^{\sqrt{(n_0-2)}} y\,dx + k\left[\{-\sqrt{n_0}+\sqrt{(n_0-2)}+\sqrt{(n_0+2)}-\sqrt{n_0}\}+\{-\sqrt{(n_0+4)}+\text{etc.}\}\right.$$
$$\left.+\text{etc.}\ldots\right]$$

$$= \int_0^{\sqrt{(n_0-2)}} y\,dx - k\,\{n_0^{-\frac{3}{2}}+(n_0+4)^{-\frac{3}{2}}+\ldots\}$$

$$= \int_0^{\sqrt{(n_0-2)}} y\,dx - \frac{k}{8}\,\{\text{N}^{-\frac{3}{2}}+(\text{N}+1)^{-\frac{3}{2}}+\ldots\}$$

$$= \int_0^{\sqrt{(n_0-2)}} y\,dx - \frac{k}{8}\sum_{\text{N}}^{\infty} n^{-\frac{3}{2}}$$

$$= \int_0^{\sqrt{(n_0-2)}} y\,dx - \frac{k}{8}\left(\sum_1^\infty n^{-\frac{3}{2}} - \sum_1^{\text{N}-1} n^{-\frac{3}{2}}\right).$$

If $n_0 = 28$, $\text{N} = 7$, $(n-2) = 26$, $\int_0^{\sqrt{26}} y\,dx = 0{\cdot}5622$ and $\sum_1^6 n^{-\frac{3}{2}} = 1{\cdot}83$.

$\sum_1^\infty n^{-\frac{3}{2}}$ can be estimated as 2·61 by the method of approximate summation of series by differential coefficients.*

Hence $$\int_0^\infty y\,dx = 0{\cdot}5622 - \tfrac{1}{8}\times 0{\cdot}6366\,(2{\cdot}61-1{\cdot}83) = 0{\cdot}500.$$

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* Boole, 'Finite Differences,' p. 90.